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BITTOR APPROACH TO THE REPRESENTATION AND PROPAGATION OF UNCERTAINTY IN MEASUREMENTS

F.Ponci[1], J.E.Johnson[2]
[1]Department of Electrical Engineering, University of South Carolina
Columbia, SC (USA)
Phone: +1-803-777-7365, Fax: +1-803-777-8045, Email: ponci@engr.sc.edu
[2]Department of Physics and Astronomy, University of South Carolina
Columbia, SC (USA)
Phone:+1- 803-777-6431, Email: jjohnson@sc.edu

Abstract *– In this paper, the authors propose a new method for the propagation of uncertainty through non linear algorithms that may contain conditional statements. The approach is based on bittors, that are bit vectors where bits are expressed in terms of their probability to take value 1 or 0. Provided the logic operations between bittors are defined, the system of bittor numbers is introduced together with the fundamental operations. The bittor numbers can be processed with any algorithm provided the fundamental operations are redefined for bittors. This approach is suitable for multithread algorithms, thus conditional statements, can be handled easily. The implementation of bittor numbers and their operations in the Matlab environment is presented together with the numerical results of an example of application. The entropy as a measure of the information content of the bittor number is defined and proposed as a metric of loss of information content due to the elaboration. The authors discuss strengths, weaknesses and challenges of the approach and provide an overview of the potential benefits of this method.*

Keywords *– uncertainty, decision-making, nonlinear functions*

I. INTRODUCTION

The representation and processing of uncertain values and the propagation of uncertainty through non-linear measurement functions and more in general algorithms is a topic of great interest among researchers. In particular, the problem of uncertainty propagation through algorithms with conditional statements is of difficult solution.

Numerical approaches based on Monte Carlo procedure and consistent with the ISO Guide to the Expression of Uncertainty in Measurement (GUM) [1], , have been adopted to assess the uncertainty due to DSP algorithms, [2] [3] [4]. Still, this approach does not satisfactorily address the issue of conditional statements, in that decisions can be opposite for values of the variables close to the threshold and the Monte Carlo analysis does not provide insight on how to control this behavior. Furthermore, the Monte Carlo methods are not suitable for on-line applications.

The unscented transform, a more efficient approach, has been introduced in [5] and extended in [6], although, also for this method, capable of dealing with the nonlinearity of the algorithm, the conditional statements preset a challenge.

The application of non-numerical "white box" approach to the propagation of uncertainty in DSP algorithms is presented in [14], and the application in presence of conditional statements is presented in [13]. This approach is suitable for the off-line analysis of algorithms.

Non numerical methods, that are more comprehensive now well developed, have been proposed for the representation and handling of measurement uncertainty and that go beyond some of the restrictions of the GUM, and that yield a computational burden that is, in some conditions, lighter than the Monte Carlo approaches. Also, some of these methods can be used for run time calculations. Among these methods, the random fuzzy variable approach, within the framework of the theory of evidence, [7][8][9], provides a tool for dealing with non random effects and a compact representation of measurement values with their uncertainty. Also among these methods, the polynomial chaos (PC) approach provides means for the compact representation of probability densities of the values of the variables [10]. PC has also been used for modeling and simulation of uncertain systems, propagating uncertainty through linear and non linear algorithms [11], and for control design [12], allowing the on-line estimation and control of the impact of load uncertainty in a power electronic system.

The purpose of this paper is to present the implementation and use, advantages and limitations of the new bittor approach to uncertainty propagation in numerical elaboration algorithms. This approach is suitable and immediately applicable in computational environments such as Matlab. Due to its nature, is also ideal for algorithms executed in embedded microproessors.

The representation and elaboration of uncertain data in form of bittors and bittor numbers naturally propagates uncertainty during the execution of the algorithm, it allows for straightforward propagation through conditional statements with clear decision making criteria, and allows for the evaluation of information content at any step in the calculation.

The mathematical foundation of bittors, the bittor algebra and the bittor numbers are presented in paragraph II, the implementation of part of the bittor features is presented in paragraph III. An overview of the application, including a numerical example is presented in paragraph V. Since the investigation of the bittor approach is in its early stages, a critical perspective and the inquiry directions that should be followed to fully assess the application feasibility and value are presented in paragraph V. Finally the conclusions are summarized in paragraph VI.

II. THE BITTOR

In this work, the authors introduce the representation and elaboration of measurement results in binary form based on the generalization of the bit of information to the bit vector, or bittor [15][17]. The bittor representation was originally conceived to address the issue of representing physical and measurable quantities in a form that is more consistent than real numbers with the limits of knowledge and measurement. While the single bit is 1 or 0, the bittor of information, can take the intermediate values, thus, the probability of actually being 1 or 0. Bittors are rooted in a firm mathematical background, in that bittors are Markov Lie monoid two-dimensional representations.

A bittor is therefore represented as (x1, x0), where x1 represetns the probability of the bittor to be 1, and x0 represents the probability to be 0, such that x1+x0=1, so a bittor that is (1, 0) represents the certain bit 1, while a bittor (0.5, 0.5) has equal probability to be a 1 or a 0. The probability can take any value between 0 and 1. Bittors can have multiple components for example to represent the probability that that two consecutive digits are both 1, or 1 and then 0, or vice versa, or both 0.

The bittor algebra, that contains the 1/0 logic as a particular case, has been defined by defining the 16 logic operations. The general structure to define the operations between bittor x=(x1, x0) and bittor y=(y1, y0) that results in bittor z=(z1, z0) and can be expressed in the form:

$$z_i = c^{\alpha}{}_{ijk} x_j y_k \qquad (1)$$

Where, i=1, 0, j, k=1, 0, α=0, 1,..., 15 and indicates the specific operation. The three dimensional tensor c takes values 1 or 0. For the AND operation, for example, c100=c101=c110=c011=0, c000=c111=c001=c010=1. Thus, the general expression:

$$z_1 = c^{\alpha}{}_{111} x_1 y_1 + c^{\alpha}{}_{110} x_1 y_0 + c^{\alpha}{}_{101} x_0 y_1 + c^{\alpha}{}_{100} x_0 y_0$$

$$z_0 = c^{\alpha}{}_{011} x_1 y_1 + c^{\alpha}{}_{010} x_1 y_0 + c^{\alpha}{}_{001} x_0 y_1 + c^{\alpha}{}_{000} x_0 y_0$$

becomes:

$$z_1 = x_1 y_1$$

$$z_0 = x_1 y_0 + x_0 y_1 + x_0 y_0$$

The 16 operations therefore correspond to 16 combinations of the products $x_j y_k$. In particular, each operation is defined by the binary four-bit code α, that defines which terms of the ordered set x_1y_1, x_1y_0, x_0y_1, x_0y_0 are contained in the expression of z_1. the remaining terms compose z_0. For example, the OR operation is characterized by α=1110.

Notice that the structure briefly introduced here comprises the classic Boolean truth tables as a particular case and adheres to the standard probability theory for the generalization for the logic operations.

The representation of bittors can be simplified just showing the upper component of the bittor.

The linear combination of bittors is closed on the bittor space, if the coefficients of the linear combination are non-negative and sum up to 1. This opens the way to defining linear combinations of logic operations, which could be used in intrinsically uncertain decision making criteria or to represent human interventions in the decision process.

The bittor numbers are binary numbers built on the bittor structure. The bittor numbers naturally carry their own uncertainty and allow for direct propagation of uncertainty through numerical algorithms, in particular algorithms containing conditional statements. An example of bittor representation of a binary number carrying no uncertainty is (1, 0)(0, 1)(1, 0), while a binary number with uncertain least significant bit, with equal probability of a 1 or a 0 is written as (1, 0)(0, 1)(0.5, 0.5), or, for brevity (1)(0)(0.5). The sequence 10.11 is thus represented as (1, 0)(0, 1).(1 0)(1 0).

The arithmetic operations between bittor numbers are carried out like the operations between binary numbers, though considering that the probabilities rather the values are combined. As a consequence, for example, the addition between two bittors results in two bittors:

$(x_1, x_0)+(y_1, y_0)=(c_1, c_0)(r_1, r_0)$

where c1 represents the probability of carry and r1 represents the probability of the result to be 1.

The bittor representation is particularly suitable for multithreaded computation. The branching point can be, for example, an *if...then...* statement. At these points two threads are spawned, each carrying the value of its own probability.

For example, suppose that the bittor number nb=(0.8, 0.2)(0.1, 0.9) enters the following *if...then...* statement to be compared with bittor number (0, 1)(1, 0):

if nb > (0)(1)
then ...

The comparison between (0.8, 0.2) and (0, 1) results in a 0.8 probability that the conditional statement is true. Thus, the computational thread following the true option carries its own probability of 0.8.

These threads can be annihilated during the elaboration, when their probability falls below a preset threshold.

Finally, the information content of a bittor has been defined through an extension of Shannon entropy and, furthermore, the information content of an entire bittor number is defined as the sum of information content of the bittors that compose it. This is an immediate way to quantify the information content of data, at any stage of the elaboration.

The expression of the information content of a bittor is Renyi's form of Shannon entropy [16]:

$$I = \log_2\left(2\left(x_1^2 + x_0^2\right)\right) \qquad (2)$$

Notice that this quantity is always equal to 1 for bittors that represent deterministically known bits, while it is less than one for uncertain bits.

In general, for a N-dimensional bittor, I will take the form:

$$I = \log_2\left(N\sum_{i=1}^{N}\left(x_i^{\,2}\right)\right) \qquad (3)$$

Finally, for a bittor number, I equals the sum of the information of all bittors.

$$I = \log_2\left(\sum_{i=1}^{N} 2\cdot\left(x_{1_i}^{\,2} + x_{0_i}^{\,2}\right)\right) \qquad (4)$$

III. IMPLEMENTATION

The bittor and bittor number representation and operations have been implemented in Matlab (at present state only part of the set has been implemented). The bittor number class has been defined, the display and support functions and the overload operations have been implemented.

The bittor number ca be defined to comprise any number of bittors, in particular, in the present implementation each bittor number is made of 8 bittors. Warning messages are provided in case of overflow. The bittor numbers can have a decimal bittors. The probabilities are represented and processed as integers, and the user can choose the base. In the present implementation the base is 256, so each element of a bittor is in fact an 8 bit integer. Notice that the number of bittors that constitute the bittor number (8 in this case) and the base of the integer representation (256) are in fact independent, although the most convenient match can be chosen.

Thanks to this framework, any algorithm that involves the basic operations can be executed with bittor number inputs with minimal modifications. This feature can be used to test the impact on measurement algorithms on uncertainty and information content.

IV. APPLICATIONS

A first example of application in form of sum of two uncertain values, A and B, is reported here. A and B are values with uniform distribution within the range 5-0.25 and 5+0.25 and 3-0.25 and 3+0.25 respectively. Thus, the values A and B are known as

$A = 5 \pm 0.25,\ B = 3 \pm 0.25,$

that in bittor form is:

$A = (1)(0)(1).(0)(0.5),\ B = (1)(1).(0)(0.5),$

Where two parts of the bittor number can be clearly identified: a perfectly known part, made of ones and zeros, and an uncertain part, with equal probability to be one or zero, expressed by the 0.5 bittor.

Thus in terms of bittor integers, A and B are:

$A = (255)(0)(255).(0)(128),\ B = (255)(255).(0)(128)$

The sum C=A+B, in terms of bittor integers is:

$C = (245)(5)(2)(0).(64)(128)$

that is:

$C = (0.957)(0.019)(0.007)(0).(0.25)(0.5)$

With this approach, the uncertainty propagates to all the bittors in the bittor number. The propagation is limited by the resolution of the representation of the probability of the individual bittor.

The PDF of C can be reconstructed from the bittor number representation and results in what shown in Figure 1.

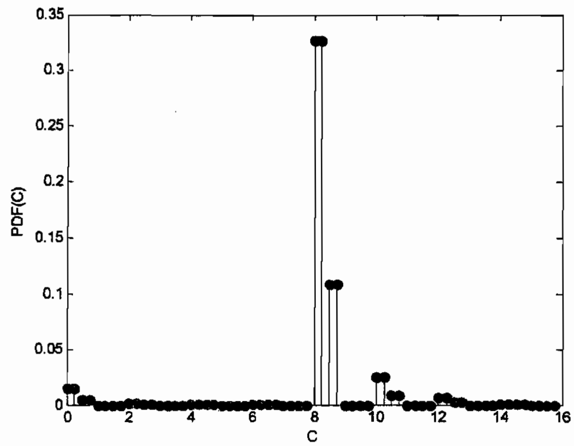

Figure 1: PDF of bittor number

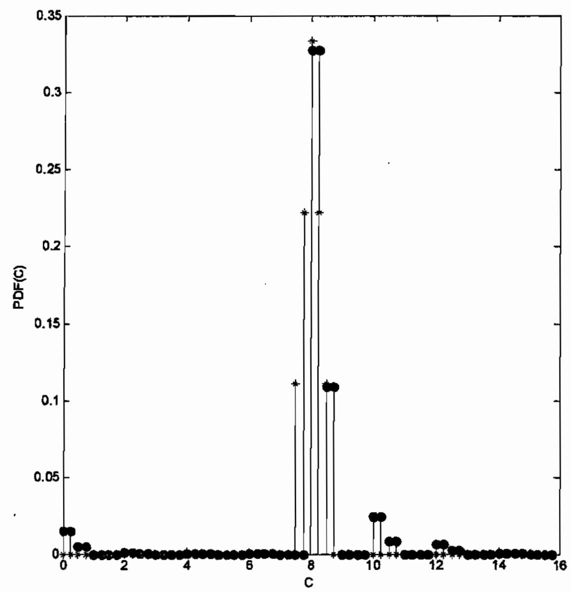

Figure 2: PDF of result of the sum as a bittor number and as reconstructed from PDFs of sum terms

Due to the uncorrelation of individual bittors in the bittor number, the PDF seems to allow for results of the addition that are not possible. The bittor number representation tends to spread out the PDF.

A simple example is introduced to illustrate the relationship between bittor numbers with multi-dimensional bittors and with two dimensional bittors.

Consider the following two bittor numbers: A=(1)(0)(0.5) and B=(1)(0)(0.5). The sum NBS of A and B can be written in terms of multidimensional bittors as:

$$NBS = A + B = (1)(0)\begin{pmatrix} x_{11} \\ x_{10} \\ x_{01} \\ x_{00} \end{pmatrix} = (1)(0)\begin{pmatrix} 0 \\ 0.25 \\ 0.5 \\ 0.25 \end{pmatrix}$$

This results means that there is 0 probability of NBS=1011 (11 in decimals), 0.25 probability of NBS=1010 (10 in decimals), 0.5 probability of NBS=1001 (9 in decimals) and 0.25 probability of NBS=1000 (8 in decimals). This multidimensional bittor representation carries along the probability of each of the possible bit configurations, so it carries the complete information. Notice that the sum of the bittor elements is equal to 1, as it should, therefore this representation has three degrees of freedom in this sample case. The PDF of NBS reconstructed from the multi dimensional bittor number is represented in Figure 3 and it is consistent with what expected.

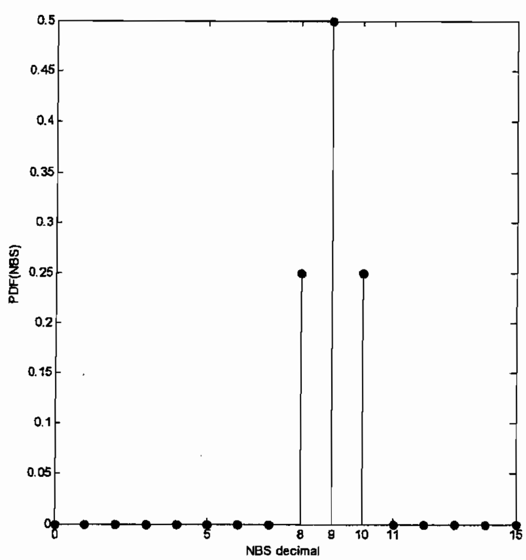

Figure 3: PDF of multi dimensional bittor number

If, instead of the multi dimensional bittor, the two dimensional bittors are used, the result of the sum, NBS', is:

$$NBS' = (1)(0)(0.25)(0.5)$$

The PDF of NBS' is reconstructed by interpreting each bittor as the probability that the corresponding power of two is added to the decimal value of NSB'. The resulting PDF is represented in Figure 4. Notice that the procedure to reconstruct this PDF can be structured as a series of convolutions, thus indicating that the reconstruction can be efficiently implemented in form of a cascade of digital FIR filters. Something else to be noticed is that this PDF is not exact, and, although similar, it does not match the one in Figure 3. In particular, this PDF is more spread out and it extends its tail to the decimal value of 11, which cannot be reached. The cause of this effect is the total uncorrelation between the two dimensional bittors that compose NBS, which, instead are correlated, being the result of the addition A+B. This spreading of the PDF is limited by the finite resolution of the value of each bittor.

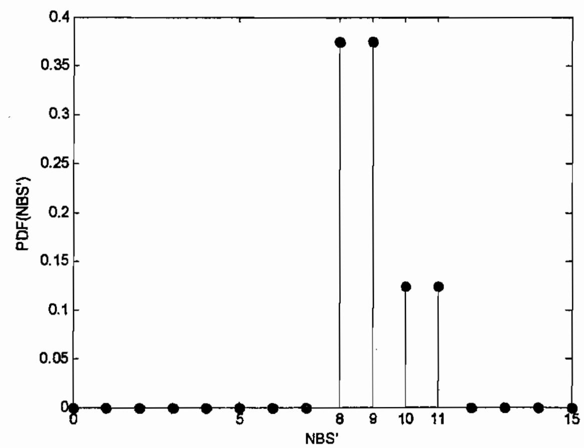

Figure 4: PDF of the result of the sum reconstructed from two dimensional bittor representation

Notice that another approach, based on the conservation of the expected value and the variance, and possible of higher order moments, could be adopted for reconstructing the PDF of NBS'. Such approach, presented here for sake of comparison, in this specific case would lead, on one hand to zero probability for the impossible value of 11, consistent with the exact PDF, and a rearrangement of the rest of the PDF as in Figure 5.

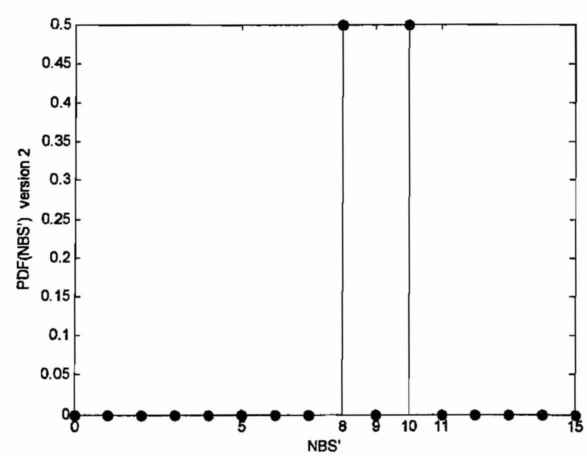

Figure 5: PDF of NBS', version 2 reconstruction

The information content of the NBS, computed, depending of the format, with equations (2)-(4), is equal to I = 2.585, while the information content of NBS' is I'=2.459. Notice that, a deterministic number with the same number of bits would carry information content I = 3. These results show, as expected, that the uncertain representation has less information that the deterministic representation and that the two dimensional bittor representation, that leads to a spreading of the PDF, has less information that the representation with the multi-dimensional bittors. The information content computed in this way can be used a quantitative metric for the information loss that occurs with the processing of uncertain data and the processing in a final resolution environment.

V. FUTURE WORK

The bittor approach requires an effort of reconciliation, wherever possible, with the expression of uncertainty of the GUM. This involves translating the standard uncertainty of a measurement in bittor, recovering the standard uncertainty from the bittor results and providing the confidence level, if at all possible. The assumption made in the most basic version of bittor numbers is that the individual bittors are not correlated. This is actually not the case when bittor numbers are the result of operation performed on other bittor numbers. In such case, the PDF of the resulting bittor number carries some error. Furthermore, even when a bittor number is not the result of numerical manipulation of other bittor numbers, the mapping of a the two dimentional bittor, bittor number, onto a PDF is difficult. In fact, the mapping is not bijective. The correlation between the bittors of a bittor number can be accounted for using multi-dimensional bittors; although the algebra for multi-dimensional bittor numbers still has to be fully formalized.

In principle, the bittor representation naturally allows for bi-modal distributions. A hint of this feature can be seen in the following example:

A=(1)(0.6)(0)(1).(0.5)(0.5).

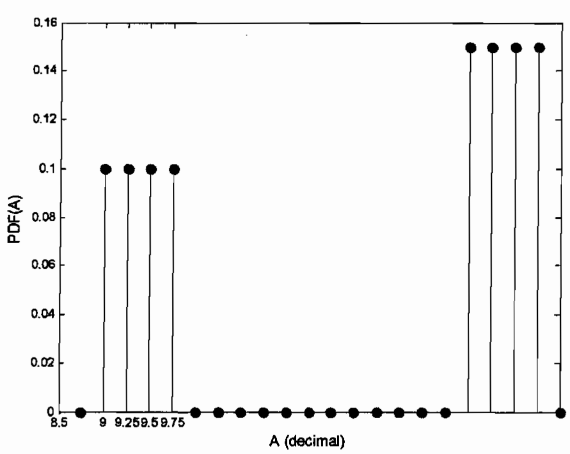

Figure 6: bimodal PDF of bittor number A above

Also, reconciliation with the other theories that go beyond the GUM must be attempted. In fact the bittor may actually support the implementation of these theories.

The information content, used as a metric of information loss, could lead to the effort of implementing algorithms in forms that minimize the loss of information. In particular, the best approach would involve the formalization of the effect of the sequence of operations and truncations to allow optimal design of the measurement and DSP algorithms processing uncertain data.

Finally, the intrinsic possibility to generate linear combinations of logic operations, such as creating a logic operation that is 70% AND and 30%OR can be greatly exploited in decision making under uncertain condition, where the uncertainty does not refer only to the data but also to the decision making laws as well.

The bittor numbers have the advantage of being perfectly fit for numerical elaboration, and can be easily implemented in embedded microprocessors. The possibility of tracking and managing the effects of truncation is an evident advantage, in particular in elaboration of integers.

VI. CONCLUSIONS

Bittors and bittor numbers have been introduced as a method to represent uncertain data and propagate the uncertainty through numerical algorithm and decision trees. The principle and mathematical background of bittors have been introduced, together with their implementation and sample applications. The gaps to be filled of a comprehensive approach to measurement uncertainty representation and propagation based on bittors have been listed, together with the strengths of the approach and the specific areas that may benefit form this new technology.

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