# Riemannian Geometry as a Generalized Heisenberg Lie Algebra

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**Abstract**

Riemannian geometry (RG) is reframed in terms of a generalized Lie algebra (GLA) with variable structure constants. We begin with an n dimensional Abelian Lie algebra of operators $X^\mu$ whose representations on a Hilbert space of square integrable functions have real simultaneous eigenvalues which can be taken as the coordinates in a real n-dimensional space $R_n$. Then n alternative independent operators $X'^\mu$ that are functions of the $X^\mu$ operators, have eigenvalues that define coordinate transformations and support the framework of contravariant and covariant tensors. We then introduce n translation operators $D^\mu$ whose exponential map is to respectively translate the $X^\mu$ as defined by the commutation rules $[D^\mu, X^\lambda] = I\, g^{\mu\nu}(X)$ where the "structure constants" are allowed to be functions of the $X^\mu$ operators. The operator "I" is to commute with all X and D operators. This gives a Riemann space when g is defined as a metric using representations of the algebra with simultaneous eigenstates of the $X^\lambda$ to define the coordinates of points in the space. In this representation, the $D^\mu$ operators have the form of generalized differential operators which, with the metric, can express the Christoffel symbols, and the Riemann, Ricci and other derivatives of the metric as commutators of the elements of the enveloping algebra. Traditional RG is obtained in the representation space of this GLA. The expression of RG as the representation of an GLA may give new insights into both areas.

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## 1. Introduction and Motivation

Lie algebras and the Lie groups which they generate have played a central role in both mathematics and theoretical physics since their introduction by Sophius Lie in 1888 [1]. Both relativistic quantum theory (QT) and the phenomenological standard model (SM) of particles and their interactions are framed in terms of observables which form Lie algebras and are firmly established [2,3,4,5]. A prime example is the Heisenberg Lie algebra (HA) among position and momentum operators which also has applications in mathematics in studies related to Fourier transforms and harmonic analysis [6,7,8,9]. But the theory of gravitation as expressed in Einstein's general theory of relativity (GR), although also firmly established, is formulated in terms of a Riemannian geometry (RG) of a curved space-time where the metric is determined by nonlinear differential equations from the distribution of matter and energy [10,11]. In GR there are no operators representing observables, and thus no commutation rules to define Lie algebras, and thus no representations of such algebras. The observables in GR are (a) the positions of events in space-time, and (b) the metric function of position in space-time (and its derivatives) which define the distance between events and thus defines the curvature of space-time. Thus QT and GR are expressed in totally different mathematical frameworks and their merger into a single theory has been a central problem in physics for over a century.

However, the space-time events in QT are the eigenvalues of the space time operators which are an essential part of the HA which also contains the Minkowsky metric which defines the distance when space-time is not curved. This led us to generalize the HA structure constants to be the Riemann metric of GR thus allowing the metric to be a function of the position operators in the algebra [12]. This generalizes the concept of a Lie algebra to allow for structure constants that are functions of the operators in the algebra and thus are no longer constants except approximately in small neighborhoods. We seek to reframe RG [13] in terms of such a generalized HA. We show that the fundamental concepts in RG such as the coordinate transformations, contravariant and covariant tensors, Christoffel symbols, Riemann and Ricci tensors, and the covariant derivative can be expressed in terms of a generalized Heisenberg Lie algebra. This investigation is reminiscent of contractions of Lie algebras where the structure constants are modified to vary smoothly among different algebras based upon certain parameters [14, 15, 16, 17, 18, 19]. In a similar way, our generalized Lie algebra allows the structure constants to be dependent upon operators in the algebra so that RG is retrieved as a representation of the algebra as one moves over the Riemann space.

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## 2. An n Dimensional Space as Eigenvectors of an Abelian Lie Algebra

Consider a set of n independent linear self adjoint operators, $X^\mu$ which form an Abelian Lie algebra of order n, where

$[X^\mu, X^\nu] = 0$ and where $\mu, \nu = 0, 1, 2, \ldots (n\text{-}1)$. (2.1)

Consider a Hilbert space of square integrable complex functions $|\Psi\rangle$ as a representation space for this algebra where a scalar product is used to normalize the vectors to unity: $\langle\Psi|\Psi\rangle = 1$.

The simultaneous eigenvectors of the Abelian Lie algebra can be written as the outer product of the $X^\mu$ eigenvectors with the notation

$$|y^0\rangle\,|y^1\rangle|y^2\rangle\ldots|y^{n-1}\rangle = |\,y^0, y^1, y^2, \ldots y^{n-1}\rangle = |y\rangle \tag{2.2}$$

where the eigenvalues $y^\mu$ label the associated eigenvectors $|y\rangle$ of the $X^\mu$ operators and where we use the notation

$$X^\mu|y\rangle = y^\mu|y\rangle \text{ where the } y^\mu \text{ are real numbers.} \tag{2.3}$$

These independent real variables $y^\mu$ can be thought of as the coordinates of an n-dimensional space $R_n$ since each set of values defines a point in the space as the eigenvectors define the points in the n-dimensional space. Let the eigenvectors be normalized to be orthonormal with the scalar product

$$\langle y_a|y_b\rangle = \delta(y^0{}_a-y^0{}_b)\,\delta(y^1{}_a-y^1{}_b)\ldots\,\delta(y^{n-1}{}_a-y^{n-1}{}_b). \tag{2.4}$$

Let the decomposition of unity

$$1 = \int_{dy}|y\rangle\langle y| \tag{2.5}$$

project the entire space onto the basis vectors $|y\rangle$ where $\langle y|$ is the dual vector to $|y\rangle$. A general vector (function) in the representation (Hilbert) space of this Lie algebra can then be written as

$$|\Psi\rangle = \int_{dy}|y\rangle\langle y|\Psi\rangle = \int_{dy}\Psi(y)\,|y\rangle, \tag{2.6}$$

where the function $\Psi(y)$ gives the "components" of the abstract vector $|\Psi\rangle$ on the basis vectors $|y\rangle$. Thus

$$\langle\Psi|\Psi\rangle = 1 = \int_{dy}\langle\Psi|y\rangle\langle y|\Psi\rangle = \int_{dy}\Psi^*(y)\Psi(y). \tag{2.7}$$

Now consider another set of n linear operators $X'^\mu$ which are independent analytic functions, $X'^\mu(X^\mu)$, of the $X^\mu$ operators also forming an Abelian Lie algebra on the same representation space for this algebra where it follows that

$$[X'^\mu, X'^\nu] = 0. \tag{2.8}$$

Let the $X'^\mu$ have eigenvectors $|y'\rangle$ and eigenvalues $y'^\mu$ given by

$$X'^\mu|y'\rangle = y'^\mu|y'\rangle \text{ where } y'^\mu \text{ are real numbers.} \tag{2.9}$$

The same orthonormality and decomposition of unity also obtain for the $|y'\rangle$ vectors which are also a complete basis for the space. Then we can let the $X'^\mu(X^\nu)$ act to the left on the dual vector $\langle y'|$ and also act to the right on the vector $|y\rangle$ as

$$\langle y'|\,X'^\mu(X^\nu)|y\rangle = \langle y'|\,X'^\mu(X^\nu)|y\rangle \text{ to give} \tag{2.10}$$

$$y'^\mu\langle y'|y\rangle = y'^\mu(y)\langle y'|y\rangle. \tag{2.11}$$

Thus the eigenvalues $y'^\mu = y'^\mu(y)$ give the transformation from the y coordinates to the y′ coordinates if the Jacobian does not vanish identically i.e. $|\partial y'^\mu/\partial y^\nu| \neq 0$ which we now require to be the case. Thus the operators $X'^\mu(X^\mu)$ define a coordinate transformation in $R_n$ from the eigenvalues (coordinates) y to the eigenvalues y′ (transformed coordinates) $R_n'$. Then the set of n real variables $y^\mu$ and the alternative variables $y'^\mu$ both can be interpreted as specifying the coordinates of points in this n-dimensional real space $R_n$ with coordinate transformations given by the functions

$$y'^\mu = y'^\mu(y). \tag{2.12}$$

It now follows that

$$dy'^\mu = (\partial y'^\mu/\partial y^\nu)\,dy^\nu \tag{2.13}$$

and any set of n functions $V^\mu(y)$ that transform as the coordinates,

$$V'^\mu(y') = (\partial y'^\mu/\partial y^\nu)\,V^\nu(y) \text{ is to be called a contravariant vector.} \tag{2.14}$$

The derivatives $\partial/\partial y^\nu$ transform as

$$\partial/\partial y'^\mu = (\partial y^\nu/\partial y'^\mu)\,\partial/\partial y^\nu \tag{2.15}$$

and any such vector $V_\mu(y)$ which transforms in this manner as

$V'_\mu(y') = (\partial y^\nu/\partial y'^\mu)\, V_\nu(y)$ is defined as a covariant vector. (2.16)

Upper indices are defined as contravariant indices while lower indices are covariant indices. Functions with multiple upper and lower indices that transform as the contravariant and covariant indices just shown are defined as tensors of the rank of the associated indices. We use the summation convention for repeated identical indices.

## 3. Translation Operators $D^\mu$ in the $X^\mu$ Space Giving a Generalized Lie Algebra

One would like to have transformations that translate one in the $R_n$ space of the operators X (and thus their eigenvalues y). We define an additional set of n operators $D^\mu$, that move one infinitesimally in each direction $y^\mu$ by using the group generated by the elements of the Lie algebra via the exponential map with transformations:

$$G(ds\,\eta) = \exp(ds\,\eta_\mu D^\mu)\ . \tag{3.1}$$

In this transformation $\eta_\mu$ is to be a unit vector in the y space and ds is to be a measure of the distance moved in the direction $\eta_\mu$ as defined below. Then

$$X'^\lambda = G\, X^\lambda\, G^{-1}. \tag{3.2}$$

By taking ds to be infinitesimal, then to one gets

$$\begin{aligned} X^\mu(s+ds) &= \exp(\, ds\,(-i/\hbar)\,\eta_\mu D^\mu\,)\, X^\lambda(s)\, \exp(-\, ds\,(-i/\hbar)\,\eta_\nu D^\nu\,) \\ &= (1 + ds\,(-i/\hbar)\,\eta_\mu D^\mu\,)\, X^\lambda(s)\ (1 - ds(-i/\hbar)\ \eta_\nu D^\nu) \\ &= X^\lambda(\,s\,) + ds\,(-i/\hbar)\,\eta_\mu\,[\,D^\mu, X^\lambda] + \text{higher order in } ds, \end{aligned} \tag{3.3}$$

where we introduced a constant $i/\hbar$ where $\hbar$ is an arbitrary real constant.

Thus the commutator $[D^\mu, X^\lambda]$ defines the way in which the transformations interact with each other in executing the transformation in keeping with the theory of Lie algebras and Lie groups. If the space is flat then there is no dependence of the commutator upon location, thus no interference among the $D^\mu$, then $[\,D^\mu, X^\lambda]$ can be normalized to $I\,\delta_\pm^{\mu\nu}$ (since $D^\mu$ is defined to translate $X^\lambda$) thus

$$[D^\mu, X^\lambda] = I\,\delta_\pm^{\mu\nu} \tag{3.4}$$

where $\delta_\pm$ is the diagonal n x n matrix with ±1 on the diagonal with off-diagonal terms zero.
This is the customary Heisenberg Lie algebra with structure constants $\delta_\pm^{\mu\nu}$ and with $[D^\mu, D^\nu] = 0$ for $\mu \neq \nu$. The operator I commutes with all elements, has a single eigenvalue $i\hbar$, and is needed to close the basis of the Lie algebra which is now of dimension 2n+1.

## 4. The Generalized 2n+1 Dimensional Lie Algebra as a Basis for Riemannian Geometry

We now wish to allow for curvature in the space $R_n$ of the X eigenvalues. Thus the [D, X] commutator is now allowed to be dependent upon the operators X and can vary from point to point in the space. We define the functions $g^{\mu\nu}(X)$ as generalized structure constants:

$$[D^\mu, X^\nu] = I\, g^{\mu\nu}(X) \qquad \text{(with the requirement that } |g| \neq 0\,) \tag{4.1}$$

with the commutators where I has the single eigenvalue $i\hbar$ and

$$[D^\mu, I\,] = 0 = [X^\mu, I\,], \text{ and } [X^\mu, X^\nu] = 0\ . \tag{4.2}$$

These (generalized structure constant) functions can also be written as

$$g^{\mu\nu}(X) = (-i/\hbar)\,[D^\mu, X^\lambda] \tag{4.3}$$

where $g^{\mu\nu}(X)$ are assumed to be analytic with $g_{\mu\nu}(X)$ defined by

$$g_{\mu\alpha}(X)\, g^{\alpha\nu}(X) = \delta_\mu{}^\nu. \tag{4.4}$$

Then using 3.3 one gets

$$X^\mu(s+ds) - X^\mu(s) = dX^\mu = ds\, i\hbar\, \eta_\mu\, g^{\mu\nu}(X). \tag{4.5}$$

We now define $g^{\mu\nu}(X)$ to be the metric for the space which gives:

$$g_{\mu\nu}(X)\, dX^\mu dX^\nu = ds^2\, g_{\mu\nu}(X)\, \eta_\alpha\, g^{\mu\alpha}(X)\, \eta_\beta\, g^{\nu\beta}(X) \tag{4.6}$$

and with $\eta_\alpha$ as a unit vector

$$g^{\alpha\beta}(X)\, \eta_\alpha \eta_\beta = 1 \text{ we get} \tag{4.7}$$

$$ds^2 = g_{\mu\nu}(X)\, dX^\mu dX^\nu \tag{4.8}$$

thus expressing the definition of the metric in customary form and defining the scalar product. This results in a 2n+1 dimensional "generalized Lie algebra" with $D^\mu$, $X^\mu$, and I as the basis elements for the algebra.

One notes that the commutator $[D^\mu, D^\nu]$ has not yet been defined. $D^\mu$, can be represented on the basis vectors of the Hilbert representation space where $X^\nu$ is diagonal as

$$\langle y|[D^\mu, X^\nu] = \langle y|\, I\, g^{\mu\nu}(X) \quad \text{with} \tag{4.9}$$

$$\langle y|\, D^\mu = (i\hbar\, g^{\mu\beta}(y)\, \partial/y^\beta + A^\mu(y))\, \langle y| = (i\hbar\, \partial^\mu + A^\mu(y)) \text{ where } \partial^\mu = g^{\mu\upsilon}(y)\, (\partial/\partial y^\nu) \tag{4.10}$$

as the representation of $D^\mu$ on the space of eigenvectors $\langle y|$ and where $A^\mu(y)$ is an arbitrary vector function of y. Note that this arbitrary vector function $A^\mu(y)$ can include other terms such as $i\hbar\, g^{\mu\beta}(y)\, \partial\Lambda(y)/y^\beta$. So one could write

$$D^\mu = D^\mu + A^\mu(X) \tag{4.11}$$

in the commutators with X as this would not alter the commutation rules of D with X. This is the most general representation of the commutation rules with the operators available using the scalar, vector and second rank tensor representations. Both the vector function $A^\mu(y)$ and the scalar function $\Lambda(y)$ could consist of multiple higher order tensor components including $g^{\mu\nu}(X)$, arbitrary scalar functions, arbitrary contravariant vector function $A^\mu(X)$ and derivatives of such objects because any contravariant vector function of the $X^\mu$ will commute with the X in the defining commutator of D and X. The $A^\mu(y)$ can even support a Yang Mills gauge transformation group, acting both on the representation space $|\Psi\rangle$, and on the $A^\mu(y)$ vector functions. In that case the $A^\mu(y)$ will have the commutation rules of that algebra with additional indices supporting Yang Mills gauge transformations. If that algebra were to be extended to include $g^{\mu\nu}(X)$ then the commutators are more complex and have not yet been explored.

Since $[D^\mu, X^\mu] = I\, g^{\mu\nu}(X)$, this is a generalization of the normal definition of a Lie algebra since $g^{\mu\nu}(X)$ is now a function of the position operators X which, in the position representation $|y\rangle$, become the eigenvalues which determine the position in the n dimensional space. Consequently, this "Lie Algebra" has "structure constants", $g^{\mu\nu}(y)$, which vary from point to point in the space. From now on we assume the general case where $g^{\alpha\beta} = g^{\alpha\beta}(y)$ is to be understood in the position representation.

In the position representation one now has

$$\langle y|\, D^\mu|\Psi\rangle = (i\hbar\, g^{\mu\upsilon}(y)\, (\partial/\partial y^\nu) + A^\mu(y)))\, \Psi(y) = (i\hbar\, \partial^\mu + A^\mu(y))\, \Psi(y) \text{ where} \tag{4.12}$$

$$\Psi(y) = \langle y|\Psi\rangle \quad \text{and} \tag{4.13}$$

$$\partial^\mu = g^{\mu\upsilon}(y)\, (\partial/\partial y^\nu) \tag{4.14}$$

and $A^\mu(y)$ is an yet undetermined vector function of $X^\nu$. In the position representation, one can write

$$g^{\mu\nu}\, (\partial/\partial y^\nu)\, \psi(y)\, \langle y| = \partial^\mu\, \psi(y)\, \langle y| = \langle y|\, (-i/\hbar)\, [D^\mu, \psi(X)] \tag{4.15}$$

for any function $\psi$ (X) allowing one to convert differential operators into commutators with $D^\mu$. It follows that $[D^\mu, [D^\nu, X^\lambda]] \neq 0$ so that the Heisenberg algebra is no longer nilpotent. Rather

$$<y| [D^\mu, [D^\nu, X^\lambda]] = (i\hbar)^2 g^{\mu\alpha} (\partial g^{\upsilon\lambda}/\partial y^\alpha) <y| \text{ since} \quad (4.16)$$

$$[A^\mu, g^{\alpha\beta}] = 0 \quad (4.17)$$

as they both are only functions of X. We have not specified the commutators $[D^\mu, D^\nu]$ yet as they are no longer zero but which in the position representation in general give

$$<y| [D^\mu, D^\nu] = [(i\hbar g^{\mu\alpha}(y) (\partial/\partial y^\alpha) + A^\mu (y)), (i\hbar g^{\upsilon\beta}(y) (\partial/\partial y^\beta)+ A^\mu (y))] <y| \quad \text{or} \quad (4.18)$$

$$<y| [D^\mu, D^\nu] = ( -\hbar^2 ( g^{\mu\alpha}(y) (\partial g^{\upsilon\beta}(y) /\partial y^\alpha) (\partial/\partial y^\beta) - g^{\nu\alpha}(y) (\partial g^{\mu\beta}(y) /\partial y^\alpha) (\partial/\partial y^\beta) + g^{\mu\alpha}(y) g^{\upsilon\beta}(y) (\partial/\partial y^\alpha) (\partial/\partial y^\beta) - g^{\nu\alpha}(y) g^{\mu\beta}(y) (\partial/\partial y^\alpha) (\partial/\partial y^\beta) ) + [D^\mu, A^\nu] + [A^\mu, A^\nu] ) <y|. \quad (4.19)$$

The third and fourth terms cancel and the last term vanishes allowing one to re-express the D commutator as $<y| [D^\mu, D^\nu] = (-\hbar^2( g^{\mu\alpha}(y)(\partial g^{\upsilon\beta}(y) /\partial y^\alpha)- g^{\nu\alpha}(y) (\partial g^{\mu\beta}(y) /\partial y^\alpha))(\partial/\partial y^\beta)+[D^\mu, A^\nu] ) <y|$. (4.20)

One can write $(\partial/\partial y^\beta) = -(i/\hbar) D_\beta$ to get (4.21)

$$[D^\mu, D^\nu] <y| = ( i\hbar B^{\mu\nu\beta} D_\beta + [D^\mu, A^\nu] ) <y| \quad (4.22)$$

$$= ( i\hbar B^{\mu\nu}{}_\beta D^\beta + [D^\mu, A^\nu]) <y| \quad (4.23)$$

But since this is true on all states $<y|$, It follows that

$$[D^\mu, D^\nu] = i\hbar B^{\mu\nu}{}_\gamma D^\gamma + [D^\mu, A^\nu] \text{ where we define} \quad (4.24)$$

$$B^{\mu\nu}{}_\gamma = ( g^{\mu\alpha}(y) (\partial g^{\upsilon\beta}(y) /\partial y^\alpha)- g^{\nu\alpha}(y) (\partial g^{\mu\beta}(y) /\partial y^\alpha) ) g_{\beta\gamma}(y) \quad (4.25)$$

and where the "structure constants" depend upon the both the metric and its derivatives. The term $[A^\mu, A^\nu]$ is zero unless $A^\mu$ contains additional operators such as with a Yang Mills gauge transformation. One also notes in the following, that since $[A^\mu, g^{\nu\alpha}(X)] = 0$, the A terms will no longer be present.

## 5. The Christoffel Symbols and Riemann and Ricci Tensors Defined

In the position diagonal representation, the Christoffel symbols are given by

$$\Gamma_{\gamma\alpha\beta} = (½) (\partial_\beta, g_{\gamma\alpha} + \partial_\alpha, g_{\gamma\beta} - \partial_\gamma, g_{\alpha\beta} ) \quad (5.1)$$

and can be written in terms of the commutators of D with the metric as

$$\Gamma_{\gamma\alpha\beta} = (½) (-i/\hbar) ( [D_\beta, g_{\gamma\alpha}] + [D_\alpha, g_{\gamma\beta}] - [ D_\gamma, g_{\alpha\beta} ] ). \quad (5.2)$$

Then using

$$g_{\alpha\beta} (X) = (-i/\hbar) [D_\alpha, X_\beta] \quad \text{one obtains} \quad (5.3)$$

$$\Gamma_{\gamma\alpha\beta} = (-½) (1/\hbar^2) ( [D_\beta, [D_\gamma, X_\alpha]] + [D_\alpha, [D_\gamma, X_\beta]] - [ D_\gamma, [D_\alpha, X_\beta] ] ). \quad (5.4)$$

The Riemann tensor becomes

$$R_{\lambda\alpha\beta\gamma} = (-i/\hbar) ( [D_\beta, \Gamma_{\lambda\alpha\gamma} ] - [D_\gamma, \Gamma_{\lambda\alpha\beta} ] ) + (\Gamma_{\lambda\beta\sigma}\Gamma^\sigma{}_{\alpha\gamma} - \Gamma_{\lambda\gamma\sigma}\Gamma^\sigma{}_{\alpha\beta} ) \quad (5.5)$$

where $\Gamma_{\gamma\alpha\beta}$ is to be inserted for the Christoffel symbols giving only commutators. One then defines the Ricci tensor as

$$R_{\alpha\beta} = g^{\mu\nu} R_{\alpha\mu\beta\nu} = (-i/\hbar) [D^\mu, X^\nu] R_{\alpha\mu\beta\nu} \text{ and also defines} \quad (5.6)$$

$$R = g^{\alpha\beta} R_{\alpha\beta} = (-i/\hbar) [D^\alpha, X^\beta] R_{\alpha\beta}. \quad (5.7)$$

where the D is not to act on the Riemann or Ricci tensor.

In general relativity the Einstein equations (5.8)

$$R_{\alpha\beta} - ½ g_{\alpha\beta} R + g_{\alpha\beta}\Lambda = (8 \pi G/c^4) T_{\alpha\beta} \text{ become}$$

$$R_{\alpha\beta} + ((i/\hbar) [D_\alpha, X_\beta]) ( ½ R - \Lambda ) = (8 \pi G/c^4) T_{\alpha\beta} \quad (5.9)$$

where $R_{\alpha\beta}$ and R are given in terms of commutators as shown above and thus all terms on the LHS consist only of commutators.

## 6. The Riemann Covariant Derivative and Fourier Transform

It is well known that the ordinary derivative of a scalar function, $V_\mu = \partial\Lambda/\partial y^\mu$, in Riemann geometry will transform under arbitrary coordinate transformations as a covariant vector. But such a derivative of a vector function of the coordinates will not transform as a tensor. The covariant derivative with respect $y^\nu$ of a contravariant vector $A^\mu$ is

$$A^\mu{}_{,\nu} = \partial A^\mu/\partial y^\nu + A^\sigma \Gamma^\mu{}_{\sigma\nu} \tag{6.1}$$

and the covariant derivative of a covariant vector $A_\mu$ is given by

$$A_{\mu,\nu} = \partial A_\mu/\partial y^\nu - A_\sigma \Gamma^\sigma{}_{\mu\nu} \tag{6.2}$$

where both $A^\mu{}_{,\nu}$ and $A_{\mu,\nu}$ transform as tensors with respect to the metric $g^{\alpha\beta}$.

One recalls for Riemannian geometry that there is a Christoffel symbol on the right hand side for each index of the tensor being differentiated. In our algebraic framework one can write the covariant differentiation of a contravariant vector $A^\mu$ as:

$$A^\mu{}_{,\nu} = i\,[D_\nu, A^\mu] + (-½)A^\sigma\,(\,[D_\nu, [D^\mu, X_\sigma]] + [D_\sigma, [D^\mu, X_\nu]] - [D^\mu, [D_\sigma, X_\nu]]\,) \tag{6.3}$$

Thus we are able to write both the regular derivative (first term) and complete it with the index contraction with the Christoffel symbol (second term). It is important to distinguish this covariant differentiation from the regular differentiation that occurs as a representation of the operator $D^\mu$ in the position representation. It follows that we can write the covariant derivative of any tensor in the same way but with a contraction of the Christoffel symbol with each of the tensor indices as is well known in Riemannian geometry.

Finally, the generalization of the Fourier transform follows from $\langle y|D^\mu|k\rangle = \langle y|D^\mu|k\rangle$ where the $D^\mu$ acts first to the left on the bra vector and then to the right on the ket vector which is to be an eigenstate of $D^\mu$ with eigenvalue $k^\mu$ giving the differential equation:

$$i\hbar\, g^{\mu\upsilon}(y)\,(\partial/\partial y^\nu)\,\langle y|k\rangle = k^\mu \langle y|k\rangle. \tag{6.4}$$

When there is no vector field $A^\mu$ present and when $g^{\mu\upsilon}$ has is constant (no y dependence), then this can be solved with:

$$\langle y|k\rangle = (2\,\pi)^{-2} \exp g_{\mu\nu}\, y^\mu k^\nu\,. \tag{6.5}$$

But in the general case with $g^{\mu\upsilon}(y)$ as a function of y this is no longer a solution and in the general case one cannot solve this equation except formally. But in a strong gravitational field near a star such as a white dwarf, one can treat the metric as the Schwarzschild solution over very small regions, such as for atomic dimensions approximately as a constant, where the radial direction can be taken as the $y^1$ direction with

$$g_{00} = 1 - r_s/y^1 \text{ and } g_{11} = -1/(1 - r_s/y^1) \tag{6.6}$$

$$\text{where } r_s = 2GM/c^2 \text{ with } g_{22} = g_{33} = -1 \tag{6.7}$$

and where G is the gravitational constant, M is the mass of the star, c is the speed of light, and $y^1$ is the distance to the center of the star.

## 7. Conclusions

It is not necessary to derive the numerous theorems that already exist in Riemannian geometry because the essential foundation is established above. If the metric $g^{\mu\nu}(X)$ is a well behaved function of the operators $X^\mu$ then the same results again will be obtained. One notes that the commutators $[D^\mu, D^\nu]$ are not arbitrary and are fixed by the metric and their commutators with the $X^\mu$. Likewise the commutators among the rotation generators in this space $L^{\mu\nu} = X^\mu D^\nu - X^\nu D^\mu$ and other commutators are complex in structure but are determined from derivatives of the metric and can be used to generate associated groups of transformations such as rotations and Lorentz transformations thus generalizing the Poincare algebra. Naturally, the truly different aspect is that the metric function is defined in the enveloping algebra of the underlying algebra and the algebra does not have the same kind of closure that one normally has for a Lie algebra. If the metric functions are sufficiently smooth, then in a sufficiently small neighborhood one gets a standard Heisenberg Lie algebra with constant (but different) numerical values for the structure constants as with the Schwarzschild metric. Even among the $[D^\mu, D^\nu]$ commutators, the derivatives of the metric result in fixed values in that small neighborhood as well as for the rotation group. The system is reminiscent of the group contraction concepts introduced by E. Inonu and E. P. Wigner and subsequent work where the structure constants are dependent upon other parameters as referenced above. Since the $D^\mu$ operators generate infinitesimal translations in the Riemann space defined by the metric of the $[D, X]$ commutator, then it follows that this approach gives the framework of all groups of motions in all Riemann spaces via the exponential map. From the mathematical point of view, the linking of two domains of mathematics such as Lie algebras & groups here with the framework of Riemannian geometry, may allow each to inform the other or suggest new avenues to explore. This is especially true when one of the domains is generalized as we have done here with the structure constants of the basic Heisenberg Lie algebra. One can now ask if the framework of Lie algebras and groups tells us something new about allowable metrics and the topology of the associated Riemann spaces. Likewise does the generalization of Lie algebras give one new tools and challenges.

From the physics point of view, there are extensive implications because the metric is determined by the distribution of matter and energy as expressed in the energy momentum tensor with Einstein's equations. And the basic generalized Heisenberg algebra equation introduced here, $[D^\mu, X^\lambda] = I\, g^{\mu\nu}(X)$, tells us something specific about the fundamental nature of the universe, namely that the interference between observations of four-momentum and four-position (space time) is given by the Einstein metric along with all other resulting commutation relations. As the primary equations of motion in quantum theory are built upon the $D^\mu$ operators with the SM, it follows that observable effects will follow this assumption which offers a framework for unifying general relativity with quantum theory. With this framework one can now extend the Poincare algebra from its Heisenberg algebra component. [12, 20, 21]. It is also of interest to observe that the representation of the $D^\mu$ operator, $(i\hbar\, g^{\mu\upsilon}(y)\,(\partial/\partial y^\nu) + A^\mu(y))$ contains arbitrary vector fields $A^\mu(y)$ in a natural manner that are necessary for the SM to support Yang Mills gauge transformations.

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