On the Integration of General Relativity with Quantum Theory and the Standard Model

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Abstract

We propose (1) that the flat space-time metric that defines the traditional covariant Heisenberg algebra commutation rules of quantum theory between the four-vector position and momentum, be generalized to be the space-time dependent Riemann metric satisfying Einstein's equations for general relativity (GR), which determine the metric from the energy-momentum tensor. The metric is then a function of the four-vector position operators which are to be expressed in the position representation. This then allows one (2) to recast the Christoffel symbols, and the Riemann and Ricci tensors in Einstein's GR differential equations for the metric, as an algebra of commutation relations among the four-vector position and momentum operators (a generalized Lie algebra). This then (3) defines the structure constants of the rest of the Poincare algebra with the space-time dependent metric of general relativity tightly integrating it with quantum theory. (4) We propose that the four momentum operator be generalized (to be gauge covariant) to include the intermediate vector bosons of the standard model (SM) further generalizing this algebra of observables to include gauge observables. Then the generalized Poincare algebra, extended with a four-vector position operator, and the phenomenological operators of the non-Abelian gauge transformations of the standard model form a larger algebra of observables thus tightly integrating all three domains. (5) We develop an algebraic (operator commutator) basis for Riemannian geometry allowing one to implement all results of Riemannian geometry into this framework. Finally, (6) we propose that the metric in the covariant momentum operator be quantized as a spin two representation putting all forces on an equal footing. Ways in which this may lead to observable effects are discussed.

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1. Introduction

The integration of Einstein's General Relativity (GR) with Quantum Theory (QT) and the (evolving) Standard Model (SM) has frustrated diverse attempts over the last hundred years although each of these three theoretical structures has proved their separate validity beyond question in their respective domains of applicability. Generally speaking, QT[1,2] and the SM[3,4] are founded upon the theory of representations of an algebra of observables. Elementary particles are each described by distinct representations of the ten parameter Poincare Lie Algebra (PA) of observables ($M^{\mu\nu}$, the Lorentz algebra of three rotations and three Lorentz transformations, along with the four-momentum operators P^μ, that generate translations in space-time) and along with the Heisenberg Lie Algebra (HA) of P^μ (E/c, P^i) and X^μ (ct, X^i) forming the foundation of QT. The SM of non-Abelian Yang-Mills gauge transformations presents another Lie algebra but of non-space-time observables, the SU(3) x SU(2) x U(1) gauge group, that restrict the admissible Poincare representations, define the allowable interactions among the particles, and provide a theory of dynamical evolution along with a complex set of computational rules. The representations of a maximum set of commuting elements of these algebras are used to index the representation space, $|\alpha_1, \alpha_2, \ldots>$ of particle states and to index operators a and a^+ that annihilate and create these particles (representations) when acting on the vacuum so that $|\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2 \ldots> = \mathbf{a}^+_{\alpha 1, \alpha 2, \ldots} \mathbf{a}^+_{\beta 1, \beta 2, \ldots} |0>$.

But GR[5,6,7] lives in a totally different mathematical environment where there are no "operators" representing observables and thus no commutation rules to define an associated operator algebra. Thus there are no algebraic representations to define the state of the system as there are in QT. GR is instead founded on non-linear differential equations for the Riemannian curved metric of space-time as determined by the energy-momentum tensor. The "observables" in GR are (1) the four vector positions of events and (2) the metric which defines the distance between two events. The Christoffel symbols, with the Riemannian and Ricci tensors, are derivatives of that metric and describe the "force" of gravity as a geodesic path in this four dimensional space as formed from derivatives of that metric. While Einstein's theory of general relativity (GR) can normally be ignored in the quantum physics at small scales as described by the SM, it dominates the large scale structure of the universe. In GR the metric of space-time is curved by the presence of matter and energy as given by $R_{\alpha\beta} - ½ g_{\alpha\beta} R + g_{\alpha\beta}\Lambda = (8 \pi G/c^4) T_{\alpha\beta}$ where $R_{\alpha\beta}$ is the Ricci tensor and $T_{\alpha\beta}$ is the energy-momentum tensor. The cosmological constant Λ represents a constant value that could possibly represent the expansion of the universe associated with Dark Energy (DE). Attempts to integrate GR with the SM using a gauge group with a massless spin two particle (graviton) have not been successful.

We will first extend the PA of the $M^{\mu\nu}$ and P^μ observables to include the four-position operators, X^μ from the HA. We call this 15 parameter combined M, P, X algebra the Extended Poincare (EP) algebra as introduced in previous work by the author [8]. The foundation of QT is the HA which already represents the four vector of position as an operator and has a Minkowsky metric $g_{\alpha\beta}$ as the structure constants. We will then generalize this metric to be a function of the position operators as the metric in GR. The representation of the four momentum operator in the position representation is the derivative which leads to the differential equations in QT. This will allow us to consider the primary observables in GR to be the four vector position operator of events and the metric which is a function of these four vector position operators. With this approach, the differential equations of GR can be recast as commutators with the four momentum operators and lead to the algebraic form for Einstein's equations that we seek. So this proposal is to generalize the flat space-time metric in the HA to be a function of the four vector position operators in the position representation, where the HA metric "structure constants" are to be the solutions to Einstein's equations derived from the generalized algebraic form of the HA. This generalization is then to be extended to the structure constants of the PA via the definition of the angular momentum four tensor defined from P and X and then extended to include the SM. We use these techniques to form an algebraic basis for Riemannian geometry (RG) [9] thus connecting our work with all general results of RG.

2. The Extended Poincare (EP) Algebra

As quantum theory is founded upon the relationship between momentum and position operations as defined in the HA, with $[X, P] = i\hbar$, and $[E, t] = i\hbar$, then a full Lie algebra of space-time observables must also include a four-position operator X^μ in order to formally include the foundations of quantum theory. This led us previously [8] to extend the PA by adjoining a four-vector position operator, X^μ ($\mu, \nu, \ldots = 0, 1, 2, 3$) whose components are to be considered as fundamental observables using a manifestly covariant form of the HA. As the X^μ generate translations in momentum, they do not generate symmetry transformations or represent conserved quantities but do provide the critical observables of space and time. We choose the Minkowski metric $g^{\mu\mu} = (+1, -1, -1, -1)$ and write the HA in the covariant form as $[P^\mu, X^\nu] = i\hbar\, g^{\mu\upsilon}\, I$ where I is an operator that commutes with all elements and has the unique eigenvalue "1" with $P^0 = E/c$, $P^1 = P_x$ ..., and $X^0 = ct$, $X^1 = x$ … . Here the operator "I" is needed to make the fifteen (15) fundamental observables in this EP algebra close into a Lie algebra with the structure constants as follows:

$$[I, P^\mu] = [I, X^\nu] = [I, M^{\mu\nu}] = 0 \tag{1a}$$

thus I commutes with all operators and has "1" as the only eigenvalue.

$$[P^\mu, X^\nu] = i\hbar\, g^{\mu\upsilon}\, I \tag{1b}$$

which is the covariant Heisenberg Lie algebra – the foundation of quantum theory.

$$[P^\mu, P^\nu] = 0 \tag{1c}$$

which insures noninterference of energy momentum measurements in all four dimensions.

$$[X^\mu, X^\nu] = 0 \tag{1d}$$

which insures noninterference of time and position measurements in all four dimensions.

$$[M^{\mu\nu}, P^\lambda] = i\hbar\, (g^{\lambda\nu} P^\mu - g^{\lambda\mu} P^\nu) \tag{1e}$$

which guarantees that P^λ transforms as a vector under $M^{\mu\nu}$.

$$[M^{\mu\nu}, X^\lambda] = i\hbar\, (g^{\lambda\nu} X^\mu - g^{\lambda\mu} X^\nu) \tag{1f}$$

which guarantees that X^λ transforms as a vector under $M^{\mu\nu}$.

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\, (g^{\mu\sigma} M^{\nu\rho} + g^{\upsilon\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\upsilon\sigma} M^{\mu\rho}) \tag{1g}$$

which guarantees that $M^{\rho\sigma}$ is a tensor under the Lorentz group generated by $M^{\mu\nu}$.

The representations of the Lorentz algebra are well-known [2,10] and are straight forward but the extension to include the four-momentum with the Poincare algebra representations are rather messy. But with our extension of the Poincare algebra to include a four-position operator, the representations are clearer. That is because as X^μ is now in the algebra, one can define the orbital angular momentum four-tensor, operator $L^{\mu\nu}$ as:

$$L^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu \text{ from which it follows that} \tag{2a}$$

$$[L^{\mu\nu}, P^\lambda] = i\hbar\, (g^{\lambda\nu} P^\mu - g^{\lambda\mu} P^\nu), \tag{2b}$$

$$[L^{\mu\nu}, X^\lambda] = i\hbar\, (g^{\lambda\nu} X^\mu - g^{\lambda\mu} X^\nu), \tag{2c}$$

$$[L^{\mu\nu}, L^{\rho\sigma}] = i\hbar\, (g^{\mu\sigma} L^{\nu\rho} + g^{\upsilon\rho} L^{\mu\sigma} - g^{\mu\rho} L^{\nu\sigma} - g^{\upsilon\sigma} L^{\mu\rho}). \tag{2d}$$

One can then define an intrinsic spin four-tensor as:

$S^{\mu\nu} = M^{\mu\nu} - L^{\mu\nu}$ with the result that (3a)

$[S^{\mu\nu}, P^{\lambda}] = 0$ (3b)

$[S^{\mu\nu}, X^{\lambda}] = 0$ (3c)

$[S^{\mu\nu}, L^{\rho\sigma}] = 0$ (3d)

$[S^{\mu\nu}, S^{\rho\sigma}] = i\hbar\,(g^{\mu\sigma} S^{\nu\rho} + g^{\upsilon\rho} S^{\mu\sigma} - g^{\mu\rho} S^{\nu\sigma} - g^{\upsilon\sigma} S^{\mu\rho})$ (3e)

Now one can separate this EP algebra into the product of two Lie algebras, the nine parameter HA (consisting of X,P,I) and the six parameter homogeneous Lorentz algebra (consisting of the $S^{\mu\nu}$). Thus one can write all EP representations as products of the representations of the two algebras. For the HA one can choose the position representation:

$X^{\mu}|\, y > = y^{\mu}|\, y >$ or the momentum representation (4a)

$P^{\mu}|\, k > = k^{\mu}|\, k >$ (4b)

or equivalently diagonalize the mass and the sign of the energy and three momenta as

$P^{\mu} P_{\mu} = m^2$, $\varepsilon(P^0)$, with eigenstates written as $|m, \varepsilon(P^0), k>$ (4c)

All representations of the homogeneous Lorentz group, $S^{\mu\nu}$, have been found by Bergmann and by Gelfand, Neimark, and Shapiro [2, 10] to be given by the two Casimir operators b_0 and b_1 defined as:

$b_0^2 + b_1^2 - 1 = ½\, g_{\mu\rho}\, g_{\nu\sigma} S^{\mu\nu} S^{\rho\sigma}$ (5a)

where $b_0 = 0, ½, 1, 3/2, \ldots(|b_1|-1)$ and where b_1 is a complex number and

$b_0\, b_1 = -¼\ \varepsilon_{\mu\nu\rho\sigma} S^{\mu\nu} S^{\rho\sigma}$ (5b)

with the rotation Casimir operator as S^2 which has the spectrum s(s+1) with the total spin

$s = b_0, b_0+1, \ldots, (|b_1| - 1)$ (5c)

and the z component of spin:

$\sigma = -s, -s+1, \ldots .s-1, s$ (5d)

Thus the homogeneous Lorentz algebra representation can be written as $|\, b_0, b_1, s, \sigma >$ which joined with the Heisenberg algebra gives the full representation space as either

$|k^{\mu}, b_0, b_1, s, \sigma > = a^{+}_{k, b0, b1, s, \sigma}\, |0>$ for the momentum representation or (6a)

$|\, y^{\mu}, b_0, b_1, s, \sigma > = a^{+}_{y, b0, b1, s, \sigma}\, |0>$ for the position representation. (6b)

These simultaneous eigenvalues represent a maximal set of commuting observables for the EP Lie algebra that can be used to index creation and annihilation operators representing particles (fields) with the quantum numbers shown. The fundamental particles must be in the representation space of these operators and thus support this Lie algebra. The algebra of observable operators must have the particles that exist in nature as the representation space of that algebra. This algebra was instrumental in finding a fully relativistically covariant Foldy-Wouthuysen type formulation that elucidates the motion of a charged Dirac particle in an electromagnetic field [11,2].

3. The Standard Model (SM)

But while the known elementary particle states can easily be fit into this infinite array of spins and continuous masses, one has a vast overabundance of states as well as a lack of a dynamical theory of their interactions. One would like to have an algebraic structure that gives all possible particles and only those particles as representations. It is here that one imposes the additional requirements of the phenomenological Standard Model (SM) which only allows three Lorentz representations (specified by b_0 and b_1) which are the pairs of values (½ , ±3/2), (0,1) and (0,2) along with a certain spectrum of masses. Specifically these representations of the Lorentz algebra for particles in the SM are as follows: (a) a unique Dirac spin s = ½ for quarks and leptons which is given by b_0 = ½ and, b_1 = ±3/2 where the sign of b_1, $\varepsilon(b_1)$, distinguishes the representation from the conjugate representation and thus where the four states of $|b_0, b_1, s, \sigma\rangle$ can be abbreviated as $|\varepsilon(b_1), \sigma = \pm ½\rangle$. These four (spinor) states support the definition of the γ^μ matrices which result from the requirement that both representation and conjugate representation be used in order for the state to be invariant under a spatial reflection which takes one from the representation to the conjugate representation having the opposite sign of b_1 thus giving the standard Dirac theory. When b_0 = 0 then the representation is equivalent to its own conjugate and thus one does not have $\varepsilon(b_1)$. There are two such pertinent cases for bosons in the SM: (b) a unique spin s = 0 (e.g. the Higgs) which is given by b_0 = 0 and, b_1 = 1, and (c) the four-vector representation given by b_0 = 0 and, b_1 = 2, which gives both s = 0 and s = 1. Linear combinations of these four spin states (s=0, $\sigma=0$ and s = 1, $\sigma= +1, 0, -1$) can be used to form a four-vector representation which is needed for the photon (electromagnetic potential A^μ) and the W and Z vector fields as well as the gluons. The dynamical theory is introduced via the phenomenological SM which imposes the requirement that these representations also support the SU(3) x SU(2) x U(1) gauge group which mixes the observables contained in the EP algebra (X^μ ,P^ν , γ^μ , s, σ) with new gauge observables (electric charge, hypercharge, isospin, color, and flavor which currently lack a space-time origin other than position dependent phase) to account for the masses and spins of physical particles along with their strong and electroweak interactions.

4. Proposed Method of Integrating GR with QT:

We seek a formulation of GR[12] as an algebra of observables in keeping with the framework of QT and the SM. We know that the "observables" in GR are values of X^ν (positions or events) and invariant lengths as determined by the Riemann metric $g^{\mu\upsilon}(X^\lambda)$ and its derivatives as functions of four-position. So the critical observable operators must be X^ν and $g^{\mu\upsilon}(X^\lambda)$, in $[P^\mu, X^\nu] = i\hbar\, g^{\mu\upsilon}(X^\lambda)$. Then the Euclidian metric that gives the HA structure constants would be generalized to be $g^{\mu\upsilon}(X^\lambda)$ written as a function of these operators. The representations of the four momentum P^μ in the Heisenberg Lie algebra in the position representation are the partial derivatives with respect to X^λ. So if we generalize the metric in the Heisenberg algebra to be the Riemann metric in Einstein's equations and express the algebra in the position representation, then the Christoffel symbols, Ricci, and Riemann tensors which are all derivatives of the metric can be rewritten as commutation relations satisfying our objective as shown below. Since the Heisenberg algebra is the foundation of QT, this would tightly integrate GR and QT into a new algebra in which the structure constants are now the Riemann metric. For this to work, we must also require that $[X^\mu, X^\nu] = 0$ to allow simultaneous measurement of the components of space-time. We call this generalized EP the "Extended Poincare Einstein" (EPE) algebra where the Riemann metric becomes the HA structure constants.

However, this EPE is not formally a Lie algebra since (1) the structure constants now vary in space-time and are operators. (2) Secondly because $g^{\mu\upsilon}(X)$ is not a member of the algebra but a function of the X operators, it has to be thought of as an analytic function of the representations of X^ν in the enveloping algebra. (3) The resulting commutators give expressions involving both the space-time dependent metric and its derivatives. The energy-momentum tensor $T_{\alpha\beta}$ is to be determined from the SM and QT. But due to the relative weakness of the gravitational interactions, $T_{\alpha\beta}$ could be approximated by a classical solution to Einstein's equations such as the Schwarzschild or Kerr solution which could determine the metric in a small local domain near a massive spherical body. That metric would then determine the local structure constants and allow one to seek the EPE Lie algebra representations in that local domain using that (locally constant) metric for the structure constants. These representations would then determine the allowable fields and the associated equations such as the modified Dirac equation for the electron in a hydrogen atom, the harmonic oscillator, and a proposed generalized uncertainty principle as described below. These could lead to observational tests.

We postulate (1) that the position operators commute with noninterfering simultaneous measurement:

$$[X^\mu, X^\nu] = 0 \tag{7a}$$

with real eigenvalues y^μ on the eigenvectors $|\,y\,\rangle$ with notation $X^\mu\,|\,y\,\rangle = y^\mu\,|\,y\,\rangle$

and (2) that the Heisenberg Lie algebra structure constants are given by the Riemann metric

$$[P^\mu, X^\nu] = i\hbar\, I\, g^{\mu\upsilon}(X) \tag{7b}$$

so we can now also write

$$g^{\mu\upsilon}(X) = (-i/\hbar)\,[P^\mu, X^\nu] \tag{7c}$$

implying that the metric tensor $g^{\mu\upsilon}(X)$ is also determined by the positon-momentum commutator as well as from Einstein's equations from the metric tensor. In the position representation one now has

$$\langle y|P^\mu|\Psi\rangle = (i\hbar\, g^{\mu\upsilon}(y)\,(\partial/\partial y^\nu) + A^\mu(y))\,)\,\Psi(y) = (\,i\hbar\,\partial^\mu + A^\mu(y))\,\Psi(y) \tag{7d}$$

$$\text{where } \Psi(y) = \langle y|\Psi\rangle \quad \text{and} \quad \partial^\mu = g^{\mu\upsilon}(y)\,(\partial/\partial y^\nu) \tag{7e}$$

and $A^\mu(X)$ is an yet undetermined vector function of X^ν which can include other vector terms such as $i\hbar\, g^{\mu\beta}(y)\,\partial\Lambda/y^\beta$ where $\Lambda(y)$ is an arbitrary scalar function.

It follows that $[P^\mu, [P^\nu, X^\rho]] \neq 0$ so that the Heisenberg algebra is no longer nilpotent. Instead one gets

$$\langle y|\,[P^\mu, g^{\alpha\beta}] = i\hbar\, g^{\mu\nu}\,(\partial g^{\alpha\beta}/\partial y_\nu)\,\langle y| = i\hbar\,\partial^\mu g^{\alpha\beta}(y)\,\langle y| \tag{8a}$$

since $[A^\mu, g^{\alpha\beta}] = 0$ as they both are functions of X

From now on $g^{\alpha\beta} = g^{\alpha\beta}(y)$ is to be understood. Thus in the position representation one can write

$$g^{\mu\nu}\,(\partial/\partial y^\nu)\;\psi(y) = \partial^\mu\,\psi(y) = -(i/\hbar)\;[P^\mu, \psi(y)] \tag{8b}$$

for any function $\psi(y)$ thus allowing one to convert differential operators into commutators with P^μ.

The momentum commutators are now no longer zero but in the position representation instead give

$$\langle y|\,[P^\mu, P^\nu] = [(i\hbar\, g^{\mu\alpha}(y)\,(\partial/\partial y^\alpha) + A^\mu(y)),\,(i\hbar\, g^{\upsilon\beta}(y)\,(\partial/\partial y^\beta) + A^\mu(y))]\;\langle y| \text{ or} \tag{8c}$$

$$\begin{aligned}\langle y|\,[P^\mu, P^\nu] = (\,-\hbar^2\,(\,&g^{\mu\alpha}(y)\,(\partial g^{\upsilon\beta}(y)/\partial y^\alpha)\,(\partial/\partial y^\beta) - g^{\nu\alpha}(y)\,(\partial g^{\mu\beta}(y)/\partial y^\alpha)\,(\partial/\partial y^\beta)\\ &+ g^{\mu\alpha}(y)\,g^{\upsilon\beta}(y)\,(\partial/\partial y^\alpha)\,(\partial/\partial y^\beta) - g^{\nu\alpha}(y)\,g^{\mu\beta}(y)\,(\partial/\partial y^\alpha)\,(\partial/\partial y^\beta)\,) + [P^\mu, A^\nu] + [A^\mu, A^\nu]\,)\,\langle y|\end{aligned} \tag{8d}$$

The third and fourth terms cancel allowing one to re-express the momentum commutator as

$$\langle y|\,[P^\mu, P^\nu] = (-\hbar^2\,(\,g^{\mu\alpha}(y)\,(\partial g^{\upsilon\beta}(y)/\partial y^\alpha) - g^{\nu\alpha}(y)\,(\partial g^{\mu\beta}(y)/\partial y^\alpha)\,)\;(\partial/\partial y^\beta) + [P^\mu, A^\nu] + [A^\mu, A^\nu]\,)\,\langle y| \tag{8e}$$

One can express

$$(\partial/\partial y^\beta) = -(i/\hbar)\,P_\beta \text{ to get} \tag{8f}$$

$$[P^\mu, P^\nu]\,\langle y| = (\,i\hbar\, B^{\mu\nu\beta}\,P_\beta + [P^\mu, A^\nu] + [A^\mu, A^\nu]\,)\;\langle y| = (\,i\hbar\, B^{\mu\nu}{}_\beta\,P^\beta + [P^\mu, A^\nu] + [A^\mu, A^\nu]\,)\;\langle y| \tag{8g}$$

But since this is true on all states $\langle y|$, It follows that

$$[P^\mu, P^\nu] = i\hbar\, B^{\mu\nu}{}_\gamma\,P^\gamma + [P^\mu, A^\nu] + [A^\mu, A^\nu] \tag{8h}$$

where we define

$$B^{\mu\nu}{}_\gamma = (\,g^{\mu\alpha}(y)\,(\partial g^{\upsilon\beta}(y)/\partial y^\alpha) - g^{\nu\alpha}(y)\,(\partial g^{\mu\beta}(y)/\partial y^\alpha)\,)\;g_{\beta\gamma}(y) \tag{8i}$$

where the "structure constants" depend upon the both the metric and its derivatives. This noncommutative four-momentum does not affect the re-expression of Einstein's equations for the metric as commutators, but does affect other EPE commutators such as for angular momentum. Non-commutativity of $[P^\mu, P^\nu]$ is already familiar in the SM for the effective momentum. Also, it is known in Riemannian geometry that the covariant derivatives do not commute. The term $[P^\mu, A^\nu]$, where A^ν is at this point an arbitrary function of the position operators, is a derivative of a vector function of position and thus is a function of position which commutes with the metric and does not alter the commutators of the Christoffel, Riemann, and Ricci tensors.

We now cast the LHS of the Einstein equations into the form of commutators of algebraic observables. Although the results look complex, they are no more so than the usual differential equations for the Christoffel, Riemann, and Ricci tensors. The Christoffel symbols

$$\Gamma_{\gamma\alpha\beta} = (½)\,(\partial_\beta, g_{\gamma\alpha} + \partial_\alpha, g_{\gamma\beta} - \partial_\gamma, g_{\alpha\beta}) \tag{9a}$$

can be written in terms of the commutators of the four-momentum with the metric as

$$\Gamma_{\gamma\alpha\beta} = (½)\,(-i/\hbar)\,([P_\beta, g_{\gamma\alpha}] + [P_\alpha, g_{\gamma\beta}] - [P_\gamma, g_{\alpha\beta}]) \tag{9b}$$

Then using $g_{\alpha\beta}(X) = (-i/\hbar)\,[P_\alpha, X_\beta]$ one obtains

$$\Gamma_{\gamma\alpha\beta} = (-½)\,(1/\hbar^2)\,([P_\beta, [P_\gamma, X_\alpha]] + [P_\alpha, [P_\gamma, X_\beta]] - [P_\gamma, [P_\alpha, X_\beta]]) \tag{9c}$$

The Riemann tensor becomes $R_{\lambda\alpha\beta\gamma} = (-i/\hbar)\,([P_\beta, \Gamma_{\lambda\alpha\gamma}] - [P_\gamma, \Gamma_{\lambda\alpha\beta}]) + (\Gamma_{\lambda\beta\sigma}\Gamma^\sigma{}_{\alpha\gamma} - \Gamma_{\lambda\gamma\sigma}\Gamma^\sigma{}_{\alpha\beta})$ (9d)

where $\Gamma_{\gamma\alpha\beta}$ is to be inserted from Eq. 9c for the Christoffel symbols giving an expression with only commutators. One then defines the Ricci tensor as:

$$R_{\alpha\beta} = g^{\mu\nu} R_{\alpha\mu\beta\nu} = (-i/\hbar)\,[P^\mu, X^\nu]\, R_{\alpha\mu\beta\nu} \tag{9e}$$

and also defines

$$R = g^{\alpha\beta} R_{\alpha\beta} = (-i/\hbar)\,[P^\alpha, X^\beta]\, R_{\alpha\beta} \tag{9f}$$

all of which must be inserted into the LHS of Einstein equations,

$$R_{\alpha\beta} - ½\, g_{\alpha\beta} R + g_{\alpha\beta}\Lambda = (8\pi G/c^4)\, T_{\alpha\beta} \tag{9g}$$

Then finally we have the LHS of Einstein equations in terms of just commutators:

$$R_{\alpha\beta} + ((i/\hbar)\,[P_\alpha, X_\beta])\,(½ R - \Lambda) = (8\pi G/c^4)\, T_{\alpha\beta} \tag{9h}$$

where $R_{\alpha\beta}$ and R are given in terms of commutators as shown above and where $T^{\alpha\beta} = \langle\Psi|\, \gamma^\alpha P^\beta + \ldots |\Psi\rangle$ with the $\gamma^\alpha P^\beta$ term being symmetrized over α and β and which are to act in both directions giving four terms for all fermions in the SM along with similar operator contributions from the boson fields and that of dark matter (DM) and where $|\Psi\rangle$ represents the fermions in the system and where $\langle\Psi|$ is understood to be the adjoint (complex conjugate times γ^0). With these substitutions, one obtains the left hand side of the Einstein equation when the equations are expressed totally in terms of the commutators of P^μ.

We do not need to expressly write out the SU(3) x SU(2) x U(1) gauge group as this is well developed in the literature. When ħ is infinitesimally small compared to the parameters of the problem, then one obtains the traditional GR equations and particle trajectories are geodesics as follows from Einstein's equations. When masses and their associated gravitational fields are small compared to the other parameters, then gravitation can be ignored and one obtains the standard Minkowsky metric of EP with the SM and the current formulation smoothly contains QT, and GR. The resulting EPE generalized Lie Algebra still has a 15 parameter basis but now with space time dependent "structure constants".

5. Proposed Integration of the SM Gauge Groups with the EPE Algebra

The principle of minimal electromagnetic interaction states that the four-momentum P^μ is to be replaced by the "effective momentum" $D^\mu = P^\mu - e\, A^\mu(X)$ where $A^\mu(X)$ is the electromagnetic four-vector potential (which is a function of space-time position) and e is the electromagnetic coupling constant. When used in the Lagrangian for a Dirac spin ½ charged particle then $\langle\Psi|\, \gamma_\mu P^\mu\, |\Psi\rangle$ is changed to $\langle\Psi|\gamma_\mu D^\mu|\Psi\rangle$ and the Dirac equation for a particle in an electromagnetic field becomes $(\gamma_\mu(P^\mu - e\, A^\mu) - m)\,|\Psi\rangle = 0$. This framework is invariant under the Abelian gauge transformations where $|\Psi\rangle \to e^{i\Lambda(X)}|\Psi\rangle$ and where $A^\mu \to A^\mu - i\partial^\mu\Lambda(x)$. The requirement of invariance under these gauge transformations was the precursor to the Yang Mills introduction of the invariance of the fundamental physical observables under non-Abelian gauge transformations which evolved into the current standard model [2,3, 4] with U(1) x SU(2) x SU(3).

We now propose (1) to further generalize the EPE algebra by replacing P^μ with the full D^μ that incorporates the mediating vector fields that serve as a representation space for the gauge groups of the standard model. Thus we now postulate that $[P^\mu, X^\nu] = i\hbar\, I\, g^{\mu\upsilon}(X)$ be altered to represent the effective four-momentum of the standard model as

$[D^\mu, X^\nu] = i\hbar\, I\, g^{\mu\upsilon}(X)$ so we can now write (10a)

$g^{\mu\upsilon}(X) = (-i/\hbar)\,[D^\mu, X^\nu]$ where in the position representation $|y\rangle$, (10b)

$\langle y|\, D^\mu = (\, i\hbar\, g^{\mu\upsilon}(y)\,(\partial/\partial y^\nu) + A^\mu\,)\,\langle y|$ (10c)

where we now define A^μ to include all vector gauge fields as:

$\langle y|\, A^\mu(X) = A^\mu(y)\,\langle y| = (-\,i g_s G_a^\mu T^a - (1/2)g'\, Y_w B^\mu - (1/2) g\, \tau_L W^\mu\,)\,\langle y|$ (10d)

thereby incorporating all of the vector bosons into the fundamental 15 parameter modified EPE algebra. In the quantum chromodynamics (QCD) sector the SU(3) symmetry is generated by T^a while G_a^μ is the SU(3) gauge field containing the gluons. These act only on the quark fields with the strong coupling constant g_s. In the electroweak sector B^μ is the U(1) gauge field and Y_w is the weak hypercharge generating the U(1) group while W^μ is the three component SU(2) gauge field and τ_L are the Pauli matrices which generate the SU(2) group (where the subscript L indicates that they only act on left fermions). The coupling constants are g' and g. There could be additional terms in the equation for D^μ that are vector functions of the position operators such as $\partial^\mu\Lambda(x)$ where $\Lambda(x)$ is an arbitrary scalar function with the XPM Lorentz representation $b_0 = 0$ and $b_1 = 1$ which gives s = 0 for the scalar field. Note that the components of the vector fields $A^\mu(y)$ are the normal particle operators for the SM with $b_0 = 0$ and $b_1 = 2$.

The gravitation metric tensor should also be composed of the $b_0 = 0$ and $b_1 = 3$ which give the mixed spin representation with s = 0 (with $\sigma = 0$), 1 (with $\sigma = 1, 0, -1$), and 2 (with $\sigma = 2, 1, 0, -1, -2$) giving a total of nine components which match the number of components (10) in a 4x4 symmetric tensor when one component is gauged away. Weinberg [1] has an in-depth discussion of the treatment of the gauge transformations for the gravitational field in chapter 10. With all mediating forces now on a essentially equal footing in the representation of the effective momentum operator D^μ, there is the possibility of incorporating gravitational gauge transformations with those of the standard model. There is also the possibility that the metric tensor could have representations of non-zero mass such as for DM which only appears to interact gravitationally with no strong or electroweak coupling.

We can now use [10b] and [10c] to write the Christoffel symbols:

$\Gamma_{\gamma\alpha\beta} = (½)\,(\partial_\beta, g_{\gamma\alpha} + \partial_\alpha, g_{\gamma\beta} - \partial_\gamma, g_{\alpha\beta}\,)$ (11a)

in terms of the commutators of the effective four-momentum with the metric as

$\Gamma_{\gamma\alpha\beta} = (½)\,(-i/\hbar)\,(\,[D_\beta, g_{\gamma\alpha}] + [D_\alpha, g_{\gamma\beta}] - [\,D_\gamma, g_{\alpha\beta}\,]\,).$ (11b)

Then using $g_{\alpha\beta}(X) = (-i/\hbar)\,[D_\alpha, X_\beta]$ one obtains

$\Gamma_{\gamma\alpha\beta} = (-½)\,(1/\hbar^2)\,(\,[D_\beta, [D_\gamma, X_\alpha]] + [D_\alpha, [D_\gamma, X_\beta]] - [\,D_\gamma, [D_\alpha, X_\beta]\,]\,)$ (11c)

The Riemann tensor becomes:

$R_{\lambda\alpha\beta\gamma} = (-i/\hbar)\,(\,[D_\beta, \Gamma_{\lambda\alpha\gamma}\,] - [D_\gamma, \Gamma_{\lambda\alpha\beta}\,]\,) + (\Gamma_{\lambda\beta\sigma}\Gamma^\sigma{}_{\alpha\gamma} - \Gamma_{\lambda\gamma\sigma}\Gamma^\sigma{}_{\alpha\beta}\,)$ (11d) where $\Gamma_{\gamma\alpha\beta}$ is to be inserted from Eq. 11c for the Christoffel symbols giving an expression with only commutators.

One then defines the Ricci tensor as

$$R_{\alpha\beta} = g^{\mu\nu} R_{\alpha\mu\beta\nu} = (-i/\hbar) [D^{\mu}, X^{\nu}] R_{\alpha\mu\beta\nu} \tag{11e}$$

and also defines

$$R = g^{\alpha\beta} R_{\alpha\beta} = (-i/\hbar) [D^{\alpha}, X^{\beta}] R_{\alpha\beta} \tag{11f}$$

all of which must be inserted into the LHS of Einstein equations,

$$R_{\alpha\beta} - ½ g_{\alpha\beta} R + g_{\alpha\beta}\Lambda = (8 \pi G/c^4) T_{\alpha\beta} \tag{11g}$$

Finally replacing the metric by the commutator we have the LHS of Einstein equations in terms of just commutators:

$$R_{\alpha\beta} + ((i/\hbar) [D_{\alpha}, X_{\beta}]) (½ R - \Lambda) = (8 \pi G/c^4) T_{\alpha\beta} \tag{11h}$$

where $R_{\alpha\beta}$ and R are given above in terms of commutators as shown. The form using the effective momentum D is the same as the form with P because the gauge fields commute with the metric.

Even with these generalizations, the generalized momentum D^{α}, still translates the operator X^{β} as

$$X'^{\beta} = \exp((-i/\hbar)\eta_{\alpha} D^{\alpha}) X^{\beta} \exp((-i/\hbar) \eta_{\alpha} D^{\alpha}) = X^{\beta} + \eta^{\beta} \tag{11i}$$

and likewise the position operator translates the D^{α} as

$$D'^{\beta} = \exp((-i/\hbar)\eta_{\alpha} X^{\alpha}) D^{\beta} \exp((-i/\hbar) \eta_{\alpha} X^{\alpha}) = D^{\beta} + \eta^{\beta} . \tag{11j}$$

However other commutators in the XPE algebra need to be recomputed using the definition $L^{\mu\nu} = X^{\mu} D^{\nu} - X^{\nu} D^{\mu}$ thus generalizing the previous equations to include both the non-commuting aspect of P and the SM generalization of P to D. One can show that

$$[L^{\mu\nu}, X^{\lambda}] = i\hbar (g^{\nu\lambda}(X) X^{\mu} - g^{\mu\lambda}(X) X^{\nu}) \tag{12a}$$

is the same form as before, and thus $L^{\mu\nu}$ still generates the Lorentz group transformations in the space X^{λ}.

But the commutator with the momentum vector is more complex:

$$[L^{\mu\nu}, P^{\lambda}] = i\hbar (g^{\lambda\upsilon}(X) P^{\mu} - g^{\lambda\mu}(X) P^{\nu}) - i\hbar (B^{\lambda\nu}{}_{\gamma} X^{\mu} - B^{\lambda\mu}{}_{\gamma} X^{\nu})P^{\gamma} . \tag{12b}$$

The intrinsic spin tensor $S^{\mu\nu}$ commutation rules, are only for the free field and have the Minkowsky metric. This means that the γ matrices and spin representations are the same as the XPM algebra. We will call the 15 parameter EPE algebra that has been extended by D^{μ} to encompass the 12 parameter SM as the 27 parameter EPESM algebra. With the three inversions (space I_s, time I_t, and particle conjugation I_c) there are 30 observables now in this algebraic structure. In a separate paper[9], we develop a foundation for Riemannian geometry based upon an operator algebra, commutation rules, and the representation space for that algebra. It is not necessary to derive all theorems for that subject but only to lay a foundation that supports the standard foundation such as found in a standard exposition [13].

6. Discussion:

Our proposed integration of GR, QT, and the SM is embodied in equations 7a, 10a-d, 11a-h, and 12a-b. Except for generalizing the metric to depend upon the space-time four vector operator, our proposal is built upon the current standard versions of the Poincare and Heisenberg Lie Algebras, the SM, and Einstein's equations for GR and all standard computational and interpretational rules for each domain. One notes that c, ħ, and G are all in the structure constants of this 27 parameter EPESM algebra on an equal footing (along with the coupling constants of the SM) and as such define the natural scale for mass, length and time commonly known as the Plank scale as well as the SM coupling constants. Specifically, all nonzero structure constants contain the metric (and/or its derivatives) for a curved space-time, $g^{\mu\upsilon}(X)$. General relativity as formulated here is both an explicit part of the phenomenological SM of interactions as well as a rich extension of the kinematic space-time infrastructure of QT whose representations are to support the allowable particles in the SM which give the complementary formulation of the strong and electroweak forces. We are exploring how the non-Abelian gauge transformations of the SM can be extended to include both the massless graviton as well as possible massive representations ($b_0 = 0$ and, $b_1 = 3$) tensor representations that could represent other states. One notes that our approach is in keeping with Einstein's view that gravitation is a result of the Riemann curvature of space-time as well as a tensor boson interaction to be incorporated with gauge forces in the SM. More specifically, we are suggesting that gravitation manifests itself through the structure constants of the fundamental observables. Nevertheless, our final result for the representation of D^μ as $D^\mu = (i\hbar\, g^{\mu\beta}(y)\, \partial/y^\beta + A^\mu(y)\,)$ puts gravity (via the metric) and the SM vector bosons on an essentially equal footing and can support computations with gravity as a classical "field" described by the metric of Einstein's equations mixed with either a classical electromagnetic field $A^\mu(y)$ or with a quantized EM field, or with a fully quantized SM of vector bosons providing the strong and electroweak forces operating in the "classical" Riemannian curved space time of Einstein, or finally with the metric taken as the spin 2 quantized field. The uncertainty relation: $\Delta x^\mu\, \Delta p^\mu \geq \hbar\, g^{\mu\upsilon}/2$ is modified so that the metric now alters the effective value of ħ both for the position-momentum and the energy-time inequalities. Currently the Λ term in Einstein's equation is the simplest explanation for dark energy (DE) and is here a pure manifestation of the GR formulation of the structure constants. Of course the question still remains of how this term arises. One notes that our approach fully integrates GR with spin and angular momentum as well as space-time.

By using those cases where Einstein's equations have been solved, one can determine $g^{\mu\upsilon}(X)$ and thus can explicitly write the structure constants for the EPESM algebra and seek its representations in neighborhoods where the metric can be considered locally constant. Thus one can consider a physical system such as a massive object like a white dwarf or neutron star with a hydrogen atom near the surface. Then the energy momentum tensor can be classically determined for the location of the object and Einstein's equations for the metric tensor can be solved for this case and inserted into the structure constants. One must then find the representations of that local Lie algebra which give the allowable states in nature which are then required to support the SM gauge groups along with the resulting altered Dirac equation. The immediate objective is to seek new predictions that issue from this algebraic framework. For a spherical (non-rotating) mass, the Schwarzschild metric has $g_{00} = (1- r_s/r)$ and $g_{rr} = -1/(1- r_s/r)$ where the Schwarzschild radius is given by $r_s = 2GM/c^2$, and r is the radius distance from the center of the spherical mass M located at r = 0 as given in spherical coordinates. This implies that $\Delta X\, \Delta P$ and $\Delta t\, \Delta E$ have effectively different values from traditional quantum theory if the gravitational field is very large. The Schwarzschild metric now gives $\Delta X_r \Delta P_r \geq (\hbar/2)\,(1/(1-r_s/r))$ and $\Delta t \Delta E \geq (\hbar/2)(1-r_s/r)$ which modify the uncertainty principle in a strong gravitational field. It follows that taking the product of these two equations that one gets $\Delta X_r\, \Delta P_r\, \Delta t\, \Delta E \geq (\hbar/2)^2$ which is independent of both r and r_s. If one considers a small region just outside of a large spherical mass then in that region one can take the z direction toward the center of the mass which gives $g_{33} = -1/(1- r_s/r)$ and $g_{00} = (1- r_s/r)$ while $g_{11} = g_{22} = -1$ and in that region the metric can be considered constant over small domains and thus effectively modify the value of ħ in the time and z direction. Thus to a good approximation, one can modify existing equations in that region using these three values of "effective ħ" in equations such as the Dirac equation. One then needs to verify "gravitationally altered effective ħ" which follows from the hypothesis that the HA metric is the Riemann metric described by Einstein. Such tests could be for radiative transitions which incur observational difficulty, or terrestrial observations but with small deviations.

8. Conclusions:

(1) Our proposal is built upon a single fundamental assumption: that the covariant Heisenberg Lie algebra $[D^{\mu}, X^{\nu}] = i \hbar I g^{\mu\nu}(X)$, with structure constants normally taken to be the Minkowsky metric, is to have $g^{\mu\nu}(X^{\lambda})$ generalized to be a function of the four-vector position operators, $g^{\mu\nu}(X)$, which in the position representation, is to be the Riemann metric as determined by Einstein's equations in general relativity. (2) This generalization then cascades throughout the Poincare algebra determining both $[D^{\mu}, D^{\nu}]$, $L^{\mu\nu} = X^{\mu} D^{\nu} - X^{\nu} D^{\mu}$ and a generalized Fourier transform with all Plank units in the structure constants of this 30 parameter algebra of observables. (3) As these commutators determine the Schrodinger, Klein Gordon, Dirac, Lagrangian, and other equations, it follows that when gravity can be ignored, then one obtains all standard QT and SM structure. But in the presence of strong gravitational fields, the Riemann metric alters the equations. (4) Einstein's equations for general relativity are reexpressed in commutator form in terms of an algebra of observable operators for space time X^{μ} and the metric $g^{\mu\nu}(X)$ in keeping with the basic concepts in QT. Thus GR, QT, and the SM are all taken over in their current form to this new larger generalized Lie algebra. (5) The representations of the gauge covariant momentum, D^{μ} in this new algebra naturally allow for the vector bosons and gauge transformations of the SM in the same expression with the Einstein metric as $\langle y|\, D^{\mu} = (i\hbar\, g^{\mu\beta}(y)\, \partial/y^{\beta} + A^{\mu}(y) + i\hbar\, g^{\mu\beta}(y)\, \partial\Lambda/y^{\beta})\, \langle y|$ thus expressing all intermediate fields together. (6) Our assumption satisfies the criterion of simplicity for Occam's Razor for this integration of accepted theories. (7) There is an immediate change in the Heisenberg uncertainty principle in strong gravitational fields as $\Delta X_r \Delta P_r \geq (\hbar/2)\, (1/(1-r_s/r))$ and $\Delta t \Delta E \geq (\hbar/2)(1-r_s/r)$ for a Schwarzschild metric as described. (8) We are seeking predictive effects in Hydrogen spectra such as near a white dwarf. Specifically, we are also seeking to understand how this proposed QT framework would alter intense gravitation behavior of particles near singularities. (9) It is important to note that if gravity is negligible, then the metric is a flat Minkowsky metric as before and all of QT and the SM are obtained. Furthermore, if one is in a large scale domain where ħ and quantum effects can be ignored then one obtains Einstein's theory of GR by replacing the $(-i/\hbar)[D,X]$ term with the metric in the position representation with D written in its differential operator form. However, in the integrated environment, $g^{\mu\nu}(y)$ alters the HA and PA commutation rules and the differential form of the covariant D operator (including the vector boson fields of the SM) and most specifically alters the Dirac equation for a hydrogen atom, the Fourier transform, and the Heisenberg uncertainty relation. Because of the emergent complexity of this system, we are using SymPy and Sage symbol manipulating programs to manage the more complex equations and derivations. Thus errors can be greatly reduced and numerical computations can be immediately executed to study the consequences of this GR-QT-SM integration and to develop a full theory with simulations and visualizations. The author is fully aware that this is only a beginning framework that now takes one to the current standard issues: (a) How to frame the gravitational gauge transformations to remove all gravitons except for the $\pm 2\hbar$ helicities while still maintaining general covariance and put all gauge transformations on an equivalent footing in D^{μ}. (b) How to correctly formulate the dynamics that extends the SM framework to include Einstein's equations where $T^{\mu\nu}$ is the source of the metric curvature. (c) How to frame general covariance for the theory and (d) how to eliminate the divergences that plague GR. The generalized Lie algebra framework proposed here does not limit the number of special dimensions to three and thus another possibility is to extend the number of special dimensions.

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